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Some non-local transformations between nonlinear diffusion equations

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Abstract. Generalized Bäcklund transformations are applied to derive links between a large number of different types of nonlinear diffusion equations, including many which are of physical significance. Some new exactly linearizable forms are determined.

1. Introduction

This paper is concerned with some extensions and further applications of a generalized Bäcklund transformation which was first introduced by Storm [1] (see also Crank [2] pp 176-7) and which transforms the nonlinear diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) \tag{1.1}$$

into the linear heat equation.

This transformation has already been applied to the more general equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) \tag{1.2}$$

which may also be written

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}(K(u)) \tag{1.3}$$

where

$$K(u)=\int_{u_0}^{u} D(\hat{u})\,\mathrm{d}\hat{u}$$

is the Kirchhoff variable, u_0 being an arbitrary constant.

Writing $u=\partial v/\partial x$, (1.2) becomes

$$\frac{\partial v}{\partial t}=D\left(\frac{\partial v}{\partial x}\right)\frac{\partial^2 v}{\partial x^2}$$

or

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left[K\left(\frac{\partial v}{\partial x}\right)\right] \tag{1.4}$$

(the arbitrary function of time which arises in integrating (1.2) may without loss of generality be set to zero by translating v by the appropriate function of time). Writing

$$v=\frac{\partial w}{\partial x}$$

(1.4) becomes (again without loss of generality)

$$\frac{\partial w}{\partial t}=K\left(\frac{\partial^2 w}{\partial x^2}\right). \tag{1.5}$$

The sequence (1.2), (1.4), (1.5) was noted by Moulana and Nariboli [3] (see also Akhatov *et al* [4, 5]).

The relevant transformation is simplest for (1.4). We introduce the hodograph type transformation

$$v=X \qquad x=V \qquad t=T \tag{1.6}$$

so that

$$\frac{\partial v}{\partial x}=\left(\frac{\partial V}{\partial X}\right)^{-1} \qquad \frac{\partial v}{\partial t}=-\frac{\partial V}{\partial T}\left(\frac{\partial V}{\partial X}\right)^{-1}$$

and

$$\frac{\partial^2 v}{\partial x^2}=-\left(\frac{\partial V}{\partial X}\right)^{-3}\frac{\partial^2 V}{\partial X^2}.$$

We then obtain

$$\frac{\partial V}{\partial T}=D^*\left(\frac{\partial V}{\partial X}\right)\frac{\partial^2 V}{\partial X^2} \tag{1.7}$$

which is an equation of the same form as (1.4), with

$$D^*(U)\equiv U^{-2}D(U^{-1}). \tag{1.8}$$

In particular when $D(u)=u^{-2}$, corresponding to (1.1), then (1.7) is the linear heat equation. This relationship was noted by Vein (see Ames [6]).

The corresponding transformation for (1.5) is (cf Akhatov *et al* [5]) the Legendre transformation

$$w=X\frac{\partial W}{\partial X}-W \qquad x=\frac{\partial W}{\partial X} \qquad t=T \tag{1.9}$$

so that

$$X=\frac{\partial w}{\partial x} \qquad V=\frac{\partial W}{\partial X}$$

with

$$\frac{\partial^2 w}{\partial x^2}=\left(\frac{\partial^2 W}{\partial X^2}\right)^{-1} \qquad \frac{\partial w}{\partial t}=-\frac{\partial W}{\partial T}.$$

This gives

$$\frac{\partial W}{\partial T}=K^*\left(\frac{\partial^2 W}{\partial X^2}\right) \tag{1.10}$$

where

$$K^*(U) \equiv -K(U^{-1}). \tag{1.11}$$

Corresponding to (1.1) this transformation maps the equation

$$\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2} = -1 \tag{1.12}$$

into the linear heat equation.

Written in terms of u, the transformation is more complicated and non-local, but (1.2) is mapped into

$$\frac{\partial U}{\partial T} = \frac{\partial}{\partial X}\left(D^*(U)\frac{\partial U}{\partial X}\right) \tag{1.13}$$

where

$$U = \frac{\partial V}{\partial X} = u^{-1}.$$

The corresponding transformation was used by Knight and Philip [7], Rosen [8] and Bluman and Kumei [9] to linearize (1.1). The more general case of mapping (1.2) into (1.13) for $D(u) = u^n$, $D^*(U) = U^{-2-n}$ was discussed by Berryman [10] and Munier *et al* [11], and the case of general $D(u)$ appears in Rogers and Shadwick [12] and Burgan *et al* [13].

More recently this transformation has been extended by Fokas and Yortsos [14] and Rosen [15] to linearize equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) + \alpha u^{-2}\frac{\partial u}{\partial x}$$

where α is a constant. This becomes

$$\frac{\partial v}{\partial t} = \left(\frac{\partial v}{\partial x}\right)^{-2}\frac{\partial^2 v}{\partial x^2} - \alpha\left(\frac{\partial v}{\partial x}\right)^{-1}$$

so that

$$\frac{\partial V}{\partial T} = \frac{\partial^2 V}{\partial X^2} + \alpha\left(\frac{\partial V}{\partial X}\right)^2 \tag{1.14}$$

and U satisfies Burgers' equation

$$\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2} + 2\alpha U\frac{\partial U}{\partial X}.$$

Equation (1.14) is transformed by the substitution $V = (1/\alpha)\ln Q$ into the linear heat equation

$$\frac{\partial Q}{\partial T} = \frac{\partial^2 Q}{\partial X^2}.$$

The purpose of this paper is to obtain further generalizations of the transformation, applying it to various problems of nonlinear diffusion type. We shall show how different classes of nonlinear equation are linked together by the transformation, and we shall derive new exactly linearizable equations. The transformations can be used to carry over results (such as exact solutions or theorems) from a given class of equations of other classes. A recent review of qualitative results for some nonlinear diffusion equations is given by Kalashnikov [16]; transforming both the equations and the conditions of the theorems under the mappings given here would give results for additional classes of equation. Some simple illustrations of transformations of exact solutions are given below.

Relationships between various of the functions arising in the analysis will usually follow that of this section, so that the following relationships will hold everywhere except section 7, and will not be continually repeated:

$$u=\frac{\partial v}{\partial x} \qquad v=\frac{\partial w}{\partial x} \qquad U=\frac{\partial V}{\partial X} \qquad V=\frac{\partial W}{\partial X}$$

$$V=x \qquad X=v \qquad T=t$$

$$W=x\frac{\partial w}{\partial x}-w \qquad w=X\frac{\partial W}{\partial X}-W \qquad U=u^{-1}.$$

The remainder of the paper is organized as follows. Section 2 is concerned with the inhomogeneous diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(f(x)u^{-2}\frac{\partial u}{\partial x}\right)$$

and the equations into which it maps, and section 3 generalizes these results to

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(f(x)D(u)\frac{\partial u}{\partial x}\right).$$

Section 4 discusses equations of the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[E\left(u,\frac{\partial u}{\partial x}\right)\right]$$

and some specific subclasses of this; assorted other examples are then considered in section 5. In section 6 transformations based on the first moment (which is related to the centre of mass) rather than on the mass are discussed. The transformations and results we obtain in sections 2-6 are necessarily rather disjointed, but many of the results are brought together and unified in the discussion of section 7. Tables 1 and 2, in particular, make clearer many of the relationships which we derive.

2. The equation $u_t=(f(x)u^{-2}u_x)_x$

2.1. Transformations

In this section we consider the inhomogeneous nonlinear diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(f(x)u^{-2}\frac{\partial u}{\partial x}\right) \tag{2.1}$$

which is a generalization of (1.1). The results we derive are related to some of those given by Munier *et al* [11].

Applying the transformations of section 1 gives

$$\frac{\partial v}{\partial t}=f(x)\left(\frac{\partial v}{\partial x}\right)^{-2}\frac{\partial^2 v}{\partial x^2} \tag{2.2}$$

so that

$$\frac{\partial V}{\partial T}=f(V)\frac{\partial^2 V}{\partial X^2}. \tag{2.3}$$

Introducing

$$A=k\int_{V_0}^{V}\frac{\mathrm{d}\hat{V}}{f(\hat{V})} \tag{2.4}$$

where V_0 and k are arbitrary constants now gives a homogeneous nonlinear diffusion equation

$$\frac{\partial A}{\partial T}=\frac{\partial}{\partial X}\left(g(A)\frac{\partial A}{\partial X}\right) \tag{2.5}$$

where $g(A)\equiv f(V)$. In the homogeneous case $f(x)=1$ we have given the standard reduction of (2.1) to the linear heat equation.

Important special cases of (2.1) arise from the radially symmetric equation

$$\frac{\partial u}{\partial t}=\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}u^{-2}\frac{\partial u}{\partial r}\right). \tag{2.6}$$

We write

$$x=N^{-N}r^{N}$$

to give

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(x^{(2N-2)/N}u^{-2}\frac{\partial u}{\partial x}\right) \tag{2.7}$$

and equation (2.3) becomes

$$\frac{\partial V}{\partial T}=V^{(2N-2)/N}\frac{\partial^2 V}{\partial X^2}. \tag{2.8}$$

Choosing V_0 appropriately and taking $k=(2-N)/N$ for $N\neq 2$ and $k=1$ for $N=2$, definition (2.4) gives

$$A=\begin{cases}V^{(2-N)/N} & N\neq 2\\ \ln V & N=2\end{cases}$$

giving for $N\neq 2$

$$\frac{\partial A}{\partial T}=\frac{\partial}{\partial X}\left(A^{(2N-2)/(2-N)}\frac{\partial A}{\partial X}\right) \tag{2.9}$$

and for $N=2$

$$\frac{\partial A}{\partial T}=\frac{\partial}{\partial X}\left(\mathrm{e}^{A}\frac{\partial A}{\partial X}\right). \tag{2.10}$$

Using well known transformations (see section 1 and [10]), equation (2.9) may in turn be mapped into

$$\frac{\partial a}{\partial t}=\frac{\partial}{\partial z}\left(a^{2/(N-2)}\frac{\partial a}{\partial z}\right) \tag{2.11}$$

so that equations of the form (2.6) with $N=N_1$ and $N=N_2$ (for N_1, $N_2 \neq 2$) may be mapped into one another if

$$\frac{2N_1-2}{2-N_1}=\frac{2}{N_2-2}$$

i.e. if $N_2=N_1/(N_1-1)$.

2.2. *Particular exact solutions*

This section is intended to be illustrative of the way in which our transformations can be applied to generate new exact solutions from known solutions. We note that the relationships derived in this paper between various classes of diffusion equation provide additional motivation for the construction of exact solutions to a given equation, since these may be mapped into solutions of other equations.

Throughout this section, β will denote an arbitrary constant. Equation (2.5) then has similarity solutions of the form given below (as noted by Waller [17], these forms may be slightly generalized by replacing X by $X+X_0$ and T by $T+T_0$ where X_0 and T_0 are arbitrary constants).

(i) $A=\Omega(X-\beta T)$. These may be determined exactly in the form

$$-\beta(X-\beta T)=\int_{A_1}^{A}\frac{g(\hat{A})}{\hat{A}+A_0}\,\mathrm{d}\hat{A}$$

where A_0 and A_1 are arbitrary constants. These map into steady-state solutions of equation (2.1):

$$u=\omega(x).$$

(ii) $A=\Omega(X/T^{1/2})$. These map into separable solutions of equations (2.1):

$$u=t^{1/2}\omega(x).$$

Equation (2.9) has additional similarity solutions of the following forms.

(i) $A=T^{(N-2)\beta}\Omega(X/T^{(1/2)-(N-1)\beta})$. These map to solutions of equation (2.6) of the form

$$u=t^{(1/2)+\beta}\omega(rt^{\beta}).$$

In particular $\beta=\frac{1}{2}$ gives the instantaneous source solution

$$A=T^{(1/2)(N-2)}\Omega(XT^{(1/2)(N-2)})$$

and the generalized dipole solution

$$u=t\omega(rt^{1/2})$$

for both of which explicit solutions can be found (see King [18]).

The choice $\beta=1/N$ gives the dipole solution

$$A=T^{(N-2)/N}\Omega(XT^{(N-2)/2N}) \tag{2.12}$$

which maps to

$$u = t^{(N+2)/2N}\omega(rt^{1/N}). \tag{2.13}$$

Solutions of the form (2.12) are available in closed form, namely

$$A = T^{(N-2)/N}\left\{-\frac{N-1}{2}\eta^2\left[1+\left(\frac{\alpha}{\eta}\right)^{2/N}\right]\right\}^{-(N-2)/2(N-1)} \tag{2.14}$$

where $\eta = XT^{(N-2)/2N}$ and α is an arbitrary constant; we note that this solution is sometimes better written as a similarity solution of the form

$$A = (-T)^{(N-2)/N}\Omega(X(-T)^{(N-2)/2N})$$

(see [18]). Expression (2.14) gives

$$v = t^{-(N-2)/2N}\lambda(xt)$$

where

$$\frac{N-1}{2}\lambda^2\left[1+\left(\frac{a}{\lambda}\right)^{2/N}\right] = -\xi^{2(N-1)/N}$$

with $\xi = xt$, and u is then given by

$$u = t^{(N+2)/2N}\frac{\mathrm{d}\lambda}{\mathrm{d}\xi}(\xi).$$

This gives a new exact solution to equation (2.6); we note that it is derived from a solution to (2.5) and not directly from (2.6).

(ii) $A = T^{(N-2)/2(N-1)}\Omega(X+\beta\ln T)$. These map to

$$u = t^{N/2(N-1)}\omega(rt^{1/2(N-1)}).$$

For $\beta = 0$ we have the separable solution to equation (2.9) and the instantaneous source solution to (2.6) both of which can be determined exactly.

(iii) $A = \exp[-\beta(N-2)T]\Omega(X/\exp[(N-1)\beta T])$. These map to solutions of the form

$$u = \mathrm{e}^{-\beta t}\omega(r/\mathrm{e}^{\beta t}).$$

Corresponding to $N = 2$ we have equation (2.10) with the following similarity solutions.

(i) $A = -2\beta\ln T + \Omega(X/T^{(1/2)-\beta})$. These map to

$$u = t^{(1/2)+\beta}\omega(rt^{\beta}).$$

(ii) $A = -\ln T + \Omega(X+\beta\ln T)$. These map to solutions

$$u = t\omega(rt^{1/2}).$$

First integrals of the similarity ordinary differential equations are available in each case, namely

$$-(\eta-\eta_0)+\beta\Omega = \mathrm{e}^{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}\eta}$$

and

$$\tfrac{1}{2}\xi^2\omega = \xi\omega^{-2}\frac{\mathrm{d}\omega}{\mathrm{d}\xi}+\beta$$

where η_0 is an arbitrary constant, β is the same in each case, and $\eta = X + \beta \ln T$, $\xi = rt^{1/2}$. The general solutions to each for $\beta = 0$ are already known [18].

(iii) $A = 2\beta T + \Omega(X/e^{\beta T})$. These map to

$$u = e^{-\beta t}\omega(r/e^{\beta t}).$$

3. The equation $u_t = (f(x)D(u)u_x)_x$

We now generalize the results of the previous section by considering general one-dimensional inhomogeneous nonlinear diffusion equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(x)D(u)\frac{\partial u}{\partial x}\right). \tag{3.1}$$

Radially symmetric homogeneous diffusion equations can easily be written in this form, as noted in the previous section.

Applying the transformations of section 1, (3.1) maps to

$$\frac{\partial v}{\partial t} = f(x)D\left(\frac{\partial v}{\partial x}\right)\frac{\partial^2 v}{\partial x^2} \tag{3.2}$$

so that

$$\frac{\partial V}{\partial T} = f(V)D^*\left(\frac{\partial V}{\partial X}\right)\frac{\partial^2 V}{\partial X^2} \tag{3.3}$$

where

$$D^*(U) \equiv U^{-2}D(U^{-1}). \tag{3.4}$$

Writing

$$A = \int_{V_0}^{V} \frac{d\hat{V}}{f(\hat{V})}$$

with V_0 an arbitrary constant gives

$$\frac{\partial A}{\partial T} = \frac{\partial}{\partial X}\left[K^*\left(g(A)\frac{\partial A}{\partial X}\right)\right] \tag{3.5}$$

where

$$K^*(U) = \int_{U_0}^{U} D^*(\hat{U})\,d\hat{U}$$

U_0 being an arbitrary constant, and

$$g(A) \equiv f(V).$$

We note that since

$$K(u) = \int_{u_0}^{u} D(\hat{u})\,d\hat{u}$$

writing $U_0 = 1/u_0$ gives

$$K^*(U) = -K(U^{-1})$$

as in section 1.

Equation (3.5) is a special case of the class of equations discussed in section 4. Writing $A=\partial B/\partial X$, equation (3.5) maps to

$$\frac{\partial B}{\partial T}=K^*\left[g\left(\frac{\partial B}{\partial X}\right)\frac{\partial^2 B}{\partial X^2}\right]. \tag{3.6}$$

Restricting attention now to the special case

$$K^*(U)=\frac{1}{n}U^n$$

equation (3.5) is

$$\frac{\partial A}{\partial T}=\frac{1}{n}\frac{\partial}{\partial X}\left[h(A)\left(\frac{\partial A}{\partial X}\right)^n\right] \tag{3.7}$$

where

$$h(A)\equiv g^n(A).$$

Equations of the form (3.7) have quite a large number of applications (see, for example, Atkinson and Jones [19] and Esteban and Vazquez [20]). Reversing the transformations we have

$$D^*(U)=U^{n-1} \qquad D(u)=u^{-n-1}$$

with

$$V=\int_{A_0}^{A} h^{1/n}(\hat{A})\,\mathrm{d}\hat{A}$$

for some constant A_0, and equation (3.1) has the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(f(x)u^{-n-1}\frac{\partial u}{\partial x}\right). \tag{3.8}$$

We have therefore derived a mapping from equation (3.7) to the more familiar equation (3.8). Furthermore, for $n=-1$ we have given a reduction from equation (3.7) to a linear diffusion equation for *any* $h(A)$. Applications of these (and other) results for equation (3.7) will be presented elsewhere.

Corresponding to equation (3.7), equation (3.6) takes the form

$$\frac{\partial B}{\partial T}=\frac{1}{n}h\left(\frac{\partial B}{\partial X}\right)\left(\frac{\partial^2 B}{\partial X^2}\right)^n$$

so that the form for $n=-1$, namely

$$\frac{\partial B}{\partial T}=-h\left(\frac{\partial B}{\partial X}\right)\left(\frac{\partial^2 B}{\partial X^2}\right)^{-1}$$

is linearizable for any $h(A)$.

A particularly common form is

$$h(A)=A^m$$

and then

$$g(A)=A^\alpha$$

with $\alpha = m/n$ and we may take

$$V = \begin{cases} \dfrac{A^{\alpha+1}}{\alpha+1} & \alpha \neq -1 \\ \ln A & \alpha = -1 \end{cases}$$

so that equation (3.8) becomes

$$\frac{\partial u}{\partial t} = \begin{cases} (\alpha+1)^{\alpha/(\alpha+1)} \dfrac{\partial}{\partial x}\left(x^{\alpha/(\alpha+1)} u^{-n-1} \dfrac{\partial u}{\partial x}\right) & \alpha \neq -1 \\ \dfrac{\partial}{\partial x}\left(\mathrm{e}^{-x} u^{-n-1} \dfrac{\partial u}{\partial x}\right) & \alpha = -1. \end{cases}$$

4. Equations of the form $u_t = [E(u, u_x)]_x$

4.1. Introduction

This section is concerned with equations which may be written in the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[E\left(u, \frac{\partial u}{\partial x}\right)\right]. \tag{4.1}$$

A number of subclasses of the form (4.1) have found physical application (see below).

Equation (4.1) becomes

$$\frac{\partial v}{\partial t} = E\left(\frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}\right) \tag{4.2}$$

and this maps to

$$\frac{\partial V}{\partial T} = E^*\left(\frac{\partial V}{\partial X}, \frac{\partial^2 V}{\partial X^2}\right) \tag{4.3}$$

an equation of the same form, with

$$E^*\left(U, \frac{\partial U}{\partial X}\right) = -UE\left(U^{-1}, -U^{-3}\frac{\partial U}{\partial X}\right). \tag{4.4}$$

Equation (4.3) then gives

$$\frac{\partial U}{\partial T} = \frac{\partial}{\partial X}\left[E^*\left(U, \frac{\partial U}{\partial X}\right)\right] \tag{4.5}$$

which is of the same form as (4.1).

In particular, equation (4.1) is mapped into itself if E may be written in the form

$$E\left(u, \frac{\partial u}{\partial x}\right) = u^{1/2}\left[F\left(u, \frac{\partial u}{\partial x}\right) - F\left(u^{-1}, -u^{-3}\frac{\partial u}{\partial x}\right)\right]$$

where F is any function.

4.2. $E(u, u_x) = D(u)u_x + f(u)$

We now consider the nonlinear convection–diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}(f(u)). \tag{4.6}$$

Equation (4.5) now has the form

$$\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(D^*(U)\frac{\partial U}{\partial X}\right)+\frac{\partial}{\partial X}(f^*(U)) \tag{4.7}$$

(i.e. it is another equation of the same class) with

$$D^*(U)=U^{-2}D(U^{-1}) \qquad f^*(U)=-Uf(U^{-1})$$

this result was given by Rogers *et al* [21]. Equation (4.7) is Burgers' equation when

$$D(u)=\alpha u^{-2} \qquad f(u)=\beta u^{-1}$$

so this case is exactly linearizable (see [14]).

4.3. $E(u, u_x)=D(u)|u_x|^{m-1}u_x$

As we have already mentioned, equations of the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(D(u)\left|\frac{\partial u}{\partial x}\right|^{m-1}\frac{\partial u}{\partial x}\right) \tag{4.8}$$

have arisen in a number of contexts. When $D(u)=1$ the equation is a special case of (1.4) corresponding to

$$K\left(\frac{\partial v}{\partial x}\right)=\left|\frac{\partial v}{\partial x}\right|^{m-1}\frac{\partial v}{\partial x}.$$

This case was discussed by Philip [22].

Equation (4.8) leads to

$$\frac{\partial v}{\partial t}=D\left(\frac{\partial v}{\partial x}\right)\left|\frac{\partial^2 v}{\partial x^2}\right|^{m-1}\frac{\partial^2 v}{\partial x^2} \tag{4.9}$$

which maps to

$$\frac{\partial V}{\partial T}=\hat{D}\left(\frac{\partial V}{\partial X}\right)\left|\frac{\partial^2 V}{\partial X^2}\right|^{m-1}\frac{\partial^2 V}{\partial X^2} \tag{4.10}$$

where

$$\hat{D}(U)=U^{1-3m}D(U^{-1}) \tag{4.11}$$

and

$$\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(\hat{D}(U)\left|\frac{\partial U}{\partial X}\right|^{m-1}\frac{\partial U}{\partial X}\right) \tag{4.12}$$

which is of the same form as equation (4.8).

We note in particular the case

$$D(u)=u^{1-3m}$$

since then

$$\hat{D}(U)=1$$

and equation (4.12) is a special case of equation (1.4). Applications of these results will also be discussed elsewhere.

4.4. Combinations of local and non-local transformations

In King [23] it was shown how simple local transformations (in particular, translations of u) may be combined with non-local transformations to map a given diffusion equation into a more general one. In this section we shall generalize the results of [23], but we start by restating them in a slightly different way. The sequence of transformations outlined in [23] is as follows (α, β, μ, ν and λ are arbitrary constants with $\alpha\nu \neq \beta\mu$, $\lambda \neq 0$).

We start with

$$\text{(a)} \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right). \tag{4.13}$$

We replace u by $(1/\alpha)[(\alpha\nu-\beta\mu)u+\mu]$ to give

$$\text{(b)} \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D\left(\frac{1}{\alpha}[(\alpha\nu-\beta\mu)u+\mu]\right)\frac{\partial u}{\partial x}\right].$$

We transform as in section 1 to give

$$\text{(c)} \qquad \frac{\partial U}{\partial T} = \frac{\partial}{\partial X}\left[U^{-2}D\left(\frac{1}{\alpha U}[(\alpha\nu-\beta\mu)+\mu U]\right)\frac{\partial U}{\partial X}\right].$$

We replace U by $\alpha U+\beta$ and T by λT to give

$$\text{(d)} \qquad \frac{\partial U}{\partial T} = \lambda\frac{\partial}{\partial X}\left[(\alpha U+\beta)^{-2}D\left(\frac{\mu U+\nu}{\alpha U+\beta}\right)\frac{\partial U}{\partial X}\right]. \tag{4.14}$$

Thus (4.13) is mapped into the more general equation (4.14); further applications of similar transformations do not lead to further increases in generality. We now generalize this result to the equation

$$\text{(i)} \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[E\left(u, \frac{\partial u}{\partial x}\right)\right]. \tag{4.15}$$

We replace u by $(1/\alpha)[(\alpha\nu-\beta\mu)u+\mu]$, x by $-(\alpha\nu-\beta\mu)x/\gamma$ and t by $\lambda(\alpha\nu-\beta\mu)^2 t/\gamma$ (α, β, μ, ν, λ and γ are arbitrary constants with $\alpha\nu\neq\beta\mu$, $\lambda\neq 0$, $\gamma\neq 0$) to give

$$\text{(ii)} \qquad \frac{\partial u}{\partial t} = -\alpha\lambda\frac{\partial}{\partial x}\left[E\left(\frac{1}{\alpha}[(\alpha\nu-\beta\mu)u+\mu], -\frac{\gamma}{\alpha}\frac{\partial u}{\partial x}\right)\right].$$

We apply the non-local transformations of section 4.1 to give

$$\text{(iii)} \qquad \frac{\partial U}{\partial T} = \alpha\lambda\frac{\partial}{\partial X}\left[UE\left(\frac{1}{\alpha U}[(\alpha\nu-\beta\mu)+\mu U], \frac{\gamma}{\alpha U^3}\frac{\partial U}{\partial X}\right)\right].$$

We replace U by $\alpha U+\beta$ to give

$$\text{(iv)} \qquad \frac{\partial U}{\partial T} = \lambda\frac{\partial}{\partial X}\left[(\alpha U+\beta)E\left(\frac{\mu U+\nu}{\alpha U+\beta}, \frac{\gamma}{(\alpha U+\beta)^3}\frac{\partial U}{\partial X}\right)\right]. \tag{4.16}$$

Hence equation (4.15) is mapped into (4.16). The detailed application of these transformations in deriving new solutions to equations of the form (4.8) will be presented elsewhere.

5. Miscellaneous examples

5.1. The equation $v_t = F(v, v_x, v_{xx})$

Here we consider the more general equation

$$\frac{\partial v}{\partial t} = F\left(v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}\right). \tag{5.1}$$

Differentiating equation (5.1) yields

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F\left(\int_{x_0}^{x} u\,\mathrm{d}x, u, \frac{\partial u}{\partial x}\right)\right] \tag{5.2}$$

for some $x_0 \equiv x_0(t)$. Equation (5.1) is mapped to

$$\frac{\partial V}{\partial T} = -\frac{\partial V}{\partial X} F\left[X, \left(\frac{\partial V}{\partial X}\right)^{-1}, -\left(\frac{\partial V}{\partial X}\right)^{-3} \frac{\partial^2 V}{\partial X^2}\right] \tag{5.3}$$

which may be differentiated to give

$$\frac{\partial U}{\partial T} = -\frac{\partial}{\partial X}\left[UF\left(X, U^{-1}, -U^{-3}\frac{\partial U}{\partial X}\right)\right]. \tag{5.4}$$

In contrast to the form (5.2), equation (5.4) is a purely local equation containing no integrals of U.

The reverse process starts with

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[G\left(x, u, \frac{\partial u}{\partial x}\right)\right] \tag{5.5}$$

which goes to

$$\frac{\partial v}{\partial t} = G\left(x, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}\right) \tag{5.6}$$

and then to

$$\frac{\partial V}{\partial T} = -\frac{\partial V}{\partial X} G\left[V, \left(\frac{\partial V}{\partial X}\right)^{-1}, -\left(\frac{\partial V}{\partial X}\right)^{-3} \frac{\partial^2 V}{\partial X^2}\right]. \tag{5.7}$$

We note that equations (5.1) and (5.7) do not contain the space variable (x or X) explicitly; equations (5.3) and (5.6) do not contain the dependent variable (v or V) explicitly.

As an illustration we consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x} + \Phi\left(\int_{-\infty}^{x} u\,\mathrm{d}x\right) u\right] \tag{5.8}$$

which is a slight generalization of the equation discussed by Nagai and Mimura [24]. This maps to

$$\frac{\partial v}{\partial t} = D\left(\frac{\partial v}{\partial x}\right)\frac{\partial^2 v}{\partial x^2} + \Phi(v)\frac{\partial v}{\partial x} \tag{5.9}$$

and then to

$$\frac{\partial V}{\partial T} = D^*\left(\frac{\partial V}{\partial X}\right)\frac{\partial^2 V}{\partial X^2} - \Phi(X) \tag{5.10}$$

where $D^*(U)=U^{-2}D(U^{-1})$. In particular, when $D(u)=u^{-2}$ then equation (5.10) is linear for *any* $\Phi(v)$. Equation (5.10) leads to

$$\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(D^*(U)\frac{\partial U}{\partial X}\right)-\frac{\mathrm{d}\Phi}{\mathrm{d}X}(X) \tag{5.11}$$

which is a standard nonlinear diffusion equation with a position-dependent sink term. Reversing the transformations, such an equation can always be mapped into one of the form (5.9).

Equation (5.9) also maps to

$$\frac{\partial w}{\partial t}=K\left(\frac{\partial^2 w}{\partial x^2}\right)+\varphi\left(\frac{\partial w}{\partial x}\right) \tag{5.12}$$

where

$$K(u)=\int_{u_0}^{u} D(\hat{u})\,\mathrm{d}\hat{u}$$

$$\varphi(v)=\int_{v_0}^{v} \Phi(\hat{v})\,\mathrm{d}\hat{v}$$

for some constants u_0 and v_0, and equation (5.12) transforms to

$$\frac{\partial W}{\partial T}=K^*\left(\frac{\partial^2 W}{\partial X^2}\right)-\varphi(X) \tag{5.13}$$

where $K^*(U)=-K(U^{-1})$.

5.2. The equation $v_t=f(v)v_{xx}+g(v)v_x+h(v)$

By writing

$$v=\int_{c_0}^{c} D(c)\,\mathrm{d}c$$

for some constant c_0, the nonlinear reaction–convection–diffusion equation

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(D(c)\frac{\partial c}{\partial x}\right)+E(c)\frac{\partial c}{\partial x}+F(c)$$

may be written in the form

$$\frac{\partial v}{\partial t}=f(v)\frac{\partial^2 v}{\partial x^2}+g(v)\frac{\partial v}{\partial x}+h(v) \tag{5.14}$$

which is a special case of equation (5.1) and maps to

$$\frac{\partial V}{\partial T}=f(X)\left(\frac{\partial V}{\partial X}\right)^{-2}\frac{\partial^2 V}{\partial X^2}-g(X)-h(X)\frac{\partial V}{\partial X} \tag{5.15}$$

so that

$$\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(f(X)U^{-2}\frac{\partial U}{\partial X}\right)-\frac{\partial}{\partial X}[g(X)+h(X)U]. \tag{5.16}$$

Hence for any $f(v)$, $g(v)$ and $h(v)$, equation (5.14) can be mapped into an equation with linear (but position-dependent) convection and reaction terms and with a very particular type of nonlinearity in the diffusion term.

5.3. *The equation* $u_t = [E(u, u_x)]_x + \alpha x u_x + \beta u$

We may slightly generalize the results of section 4.1 by considering equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[E\left(u, \frac{\partial u}{\partial x}\right)\right] + \alpha x \frac{\partial u}{\partial x} + \beta u \tag{5.17}$$

where α and β are arbitrary constants. This gives

$$\frac{\partial v}{\partial t} = E\left(\frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}\right) + \alpha x \frac{\partial v}{\partial x} + (\beta - \alpha) v$$

and

$$\frac{\partial V}{\partial T} = E^*\left(\frac{\partial V}{\partial X}, \frac{\partial^2 V}{\partial X^2}\right) + \alpha^* X \frac{\partial V}{\partial X} + (\beta^* - \alpha^*) V$$

where

$$\alpha^* = \alpha - \beta \qquad \beta^* = -\beta$$

and

$$E^*\left(U, \frac{\partial U}{\partial X}\right) = -UE\left(U^{-1}, -U^{-3}\frac{\partial U}{\partial X}\right).$$

Hence we obtain

$$\frac{\partial U}{\partial T} = \frac{\partial}{\partial X}\left[E^*\left(U, \frac{\partial U}{\partial X}\right)\right] + \alpha^* X \frac{\partial U}{\partial X} + \beta^* U \tag{5.18}$$

which is an equation of the same form as (5.17).

6. First-moment-based transformations

In this section we return to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right). \tag{6.1}$$

Equation (6.1) is a conservation law and expresses conservation of mass. Since

$$v = \int_{x_0(t)}^{x} u(x, t)\, \mathrm{d}x$$

for some $x_0(t)$, the variable v gives the total mass between x_0 and x and is the basis for the transformation described in section 1. It is important for most of our results that the equation we wish to deal with be written in the conservation form

$$\frac{\partial m}{\partial t} + \frac{\partial J}{\partial x} = 0 \tag{6.2}$$

by appropriate choices of the variables m and J. For more general descriptions of the application of non-local transformations to conservation laws see, for example, Kingston and Rogers [25], Kingston *et al* [26] and Rosen [27].

The transformation proceeds by writing

$$m = \frac{\partial n}{\partial x}$$

and integrating with respect to x, after which a hodograph-type transformation is made. A more general discussion of such transformations has been given by Clarkson *et al* [28].

In transforming equation (6.1) we have taken

$$m = u \qquad J = -D(u)\frac{\partial u}{\partial x} \qquad n = v.$$

Equation (6.1), however, may also be written in a conservation form which expresses conservation of the first moment by writing

$$\frac{\partial}{\partial t}(xu) = \frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}[K(u)] - K(u)\right]. \tag{6.3}$$

We then define

$$m = xu \qquad u = x^{-1}\frac{\partial n}{\partial x}$$

with

$$J = -x\frac{\partial}{\partial x}[K(u)] + K(u).$$

Hence $n = \int_{x_1(t)}^{x} xu\,\mathrm{d}x$ for some $x_1(t)$, giving the first moment of u between x_1 and x. We then obtain

$$\frac{\partial n}{\partial t} = x\frac{\partial}{\partial x}\left[K\left(x^{-1}\frac{\partial n}{\partial x}\right)\right] - K\left(x^{-1}\frac{\partial n}{\partial x}\right). \tag{6.4}$$

Equation (6.4) may be further integrated by writing

$$n = x\frac{\partial w}{\partial x} - w \tag{6.5}$$

to recover (without loss of generality)

$$\frac{\partial w}{\partial t} = K\left(\frac{\partial^2 w}{\partial x^2}\right).$$

The relationship

$$u = \frac{\partial^2 w}{\partial x^2}$$

holds, as in section 1, and equation (6.5) shows that

$$n = W.$$

We note also that by introducing $x = y^{-1}$ we may write equation (6.4) in the conservation form

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial y}\left[yK\left(-y^3\frac{\partial n}{\partial y}\right)\right]. \tag{6.6}$$

We now apply the hodograph-type transformation to equation (6.4), writing

$$V = x \qquad Z = n \qquad T = t$$

together with

$$C = y = V^{-1}.$$

Then

$$\frac{\partial V}{\partial T} = -V\frac{\partial}{\partial Z}\left\{K\left[V^{-1}\left(\frac{\partial V}{\partial Z}\right)^{-1}\right]\right\} + \frac{\partial V}{\partial Z}K\left[V^{-1}\left(\frac{\partial V}{\partial Z}\right)^{-1}\right] \tag{6.7}$$

and

$$\frac{\partial C}{\partial T} = \frac{\partial}{\partial Z}\left\{CK\left[-C^{3}\left(\frac{\partial C}{\partial Z}\right)^{-1}\right]\right\}. \tag{6.8}$$

Considering now the special case

$$D(u) = u^{\alpha}$$

so that

$$K(u) = \begin{cases} \dfrac{1}{\alpha+1}u^{\alpha+1} & \alpha \neq -1 \\ \ln u & \alpha = -1 \end{cases}$$

equation (6.8) is

$$\frac{\partial C}{\partial T} = \begin{cases} \dfrac{(-1)^{\alpha+1}}{\alpha+1}\dfrac{\partial}{\partial Z}\left[C^{3\alpha+4}\left(\dfrac{\partial C}{\partial Z}\right)^{-(\alpha+1)}\right] & \text{if } \alpha \neq 1 \\ -\dfrac{\partial}{\partial Z}\left[C\ln\left(-C^{-3}\dfrac{\partial C}{\partial Z}\right)\right] & \text{if } \alpha = -1. \end{cases} \tag{6.9}$$

Equation (6.9) belongs to the class discussed in section 4.3. In the special case when

$$D(u) = u^{-2}$$

so that $\alpha = -2$, we have given a mapping of (6.1) into itself.

Equation (6.8) may be integrated by noting that

$$C = \frac{\partial v}{\partial Z}$$

so that

$$\frac{\partial v}{\partial T} = \frac{\partial v}{\partial Z}K\left[-\left(\frac{\partial v}{\partial Z}\right)^{3}\left(\frac{\partial^{2} v}{\partial Z^{2}}\right)^{-1}\right]$$

and since

$$Z = W \qquad v = X$$

we may then recover

$$\frac{\partial w}{\partial T} = -K\left[\left(\frac{\partial^{2} W}{\partial X^{2}}\right)^{-1}\right].$$

Finally, we note that conserved quantities for wider classes of nonlinear diffusion equation have been given by King [29], and these can also be used as the basis for non-local transformations in a manner similar to that described in this section.

Table 1. A summary of some of the relationships derived in this paper.

		(1) $\sigma_t = \frac{1}{\alpha+1}[\zeta^{-(3\alpha+2)}(-\sigma)^{-(\alpha+1)}]_{\zeta\zeta}$	$\xleftrightarrow{\sigma=w_\zeta}$	(2) $w_t = \frac{1}{\alpha+1}[\zeta^{-(3\alpha+2)}(-w_\zeta)^{-(\alpha+1)}]_\zeta$	$\xleftrightarrow{w=\psi_\zeta}$	(3) $\psi_t = \frac{1}{\alpha+1}\zeta^{-(3\alpha+2)}(-\psi_{\zeta\zeta})^{-(\alpha+1)}$
				$\updownarrow$		$\updownarrow$ $\psi+y=w\zeta$
				(4) $\zeta_t = -\frac{1}{\alpha+1}[\zeta^{-(3\alpha+2)}(-\zeta_w)^{\alpha+1}]_w$	$\xleftrightarrow{\zeta=y_w}$	(5) $y_t = -\frac{1}{\alpha+1}y_w^{-(3\alpha+2)}(-y_{ww})^{\alpha+1}$
						$\updownarrow$
(6) $a_t = -[u^\alpha a^{-1}]_{uu}$	$\xleftrightarrow{a=y_u}$	(7) $y_t = -[u^\alpha y_u^{-1}]_u$	$\xleftrightarrow{y=p_u}$	(8) $p_t = -u^\alpha p_{uu}^{-1}$		
$\updownarrow$ $b=u^3a$, $x=1/u$		$\updownarrow$		$\updownarrow$ $p+v=uv$		
		(9) $u_t = [u^\alpha u_y]_y$	$\xleftrightarrow{u=v_y}$	(10) $v_t = v_y^\alpha v_{yy}$	$\xleftrightarrow{v=w_y}$	(11) $w_t = \frac{1}{\alpha+1}w_{yy}^{\alpha+1}$
				$\updownarrow$		$\updownarrow$ $w+z=vy$
		(12) $x_t = [x^{-(\alpha+2)}x_v]_v$	$\xleftrightarrow{x=y_v}$	(13) $y_t = y_v^{-(\alpha+2)}y_{vv}$	$\xleftrightarrow{y=z_v}$	(14) $z_t = -\frac{1}{\alpha+1}z_{vv}^{-(\alpha+1)}$
		$\updownarrow$		$\updownarrow$ $q+y=xv$		$\updownarrow$
(15) $b_t = -[x^{-(\alpha+2)}b^{-1}]_{xx}$	$\xleftrightarrow{b=v_x}$	(16) $v_t = -[x^{-(\alpha+2)}v_x^{-1}]_x$	$\xleftrightarrow{v=q_x}$	(17) $q_t = -x^{-(\alpha+2)}q_{xx}^{-1}$		
				(18) $\xi_t = \frac{1}{\alpha+1}[\xi^{3\alpha+4}(-\xi_z)^{-(\alpha+1)}]_z$	$\xleftrightarrow{\xi=v_z}$	(19) $v_t = \frac{1}{\alpha+1}v_z^{3\alpha+4}(-v_{zz})^{-(\alpha+1)}$
				$\updownarrow$		$\updownarrow$ $\varphi+v=\xi z$
		(20) $\rho_t = -\frac{1}{\alpha+1}[\xi^{3\alpha+4}(-\rho)^{\alpha+1}]_{\xi\xi}$	$\xleftrightarrow{\rho=z_\xi}$	(21) $z_t = -\frac{1}{\alpha+1}[\xi^{3\alpha+4}(-z_\xi)^{\alpha+1}]_\xi$	$\xleftrightarrow{z=\varphi_\xi}$	(22) $\varphi_t = -\frac{1}{\alpha+1}\xi^{3\alpha+4}(-\varphi_{\xi\xi})^{\alpha+1}$

7. Discussion

We start by summarizing in the form of table 1 many of the relationships we have determined; table 1 also includes links to many additional equations. The transformations which have been included are those which can be obtained without significantly increasing the complexity of the equations involved. $\alpha \neq -1$ is assumed throughout. Horizontal lines represent relationships based on integration (to the right) or differentiation (to the left). Solid vertical lines represent hodograph type transformations, dashed lines Legendre transformations, and the dashed-dotted line is a purely local transformation. In order to gain brevity the notation is rather different from most of the paper; suffices represent differentiation and in particular the t suffix in each case means differentiation with respect to t keeping the independent variable appearing in the suffix on the right-hand side of the equation fixed.

The following points about table 1 may be made.

(a) Further relationships not listed in the table exist between the various functions; in particular

$\xi = 1/y \qquad \rho = -uy^3 \qquad$ (giving a local transformation between (20) and (9))

$\zeta = 1/v \qquad \sigma = -xv^3 \qquad$ (giving a local transformation between (1) and (12))

$$z_y = uy \qquad w_v = xv$$

$$y_x = bx \qquad q_v = v/b \qquad v_u = au \qquad p_y = y/a$$

$$v_\xi = \rho\xi \qquad \varphi_z = z/\rho \qquad y_\zeta = \sigma\zeta \qquad \psi_w = w/\sigma.$$

Equations in which the right-hand side can be written exactly as a second derivative have first-moment-based transformations resulting from these relationships as follows:

between (6) and (16) by $\quad v_u = ua \quad u = 1/x$

between (15) and (7) by $\quad y_x = xb \quad x = 1/u$

between (9) and (21) by $\quad z_y = yu \quad y = 1/\xi$

between (12) and (2) by $\quad w_v = vx \quad v = 1/\zeta$

between (20) and (10) by $\quad v_\xi = \xi\rho \quad \xi = 1/y$

between (1) and (13) by $\quad y_\zeta = \zeta\sigma \quad \zeta = 1/v.$

The third and fourth of these correspond to the transformation described in section 6. Further direct links between other pairs of equations in table 1 can also be found.

(b) The transformation from (9) to (7) and (8) (or equivalently from (12) to (16) and (17)) was noted and exploited by Bouillet [30].

(c) Making the purely local transformation

$$a = (\alpha+1)u^\alpha \hat{u} \qquad u = \hat{x}^{1/(\alpha+1)}$$

equation (6) becomes

$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial}{\partial \hat{x}}\left(\hat{x}^{\alpha/(\alpha+1)}\hat{u}^{-2}\frac{\partial \hat{u}}{\partial \hat{x}}\right)$$

which provides the link with the transformations of subsection 2.1.

(d) The relationship between (4) and (10) (or equivalently (18) and (13)) corresponds to the special case referred to at the end of subsection 4.3.

(e) When $\alpha = -\frac{2}{3}$, (5) and (11) are identical so that the hodograph transformation maps (5) into itself. The same holds true for (14) and (19) with $\alpha = -\frac{4}{3}$.

Table 2 generalizes the results of table 1 by including an arbitrary function $g(y)$; $h(\xi)$ is defined by

$$h(\xi) \equiv \xi^{3\alpha+4} g(1/\xi).$$

Table 2 follows the notation of table 1 and $\alpha \neq -1$ is again assumed. We note that fewer transformations can be included without significantly increasing the complexity of the equations involved. The following points may be made about table 2.

(a) The relationships

$$v_\xi = \xi\rho \qquad \xi = 1/y$$

and

$$z_y = yu \qquad y = 1/\xi$$

among others, hold as before and give first-moment-based transformations between (8) and (2) and between (1) and (9).

(b) Introducing

$$\hat{u} = g^{1/(\alpha+1)}(y)u \qquad \hat{x} = \int^y g^{-1/(\alpha+1)}(\hat{y})\,\mathrm{d}\hat{y}$$

(1) becomes

$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial}{\partial \hat{x}}\left(f(\hat{x})\hat{u}^\alpha \frac{\partial \hat{u}}{\partial \hat{x}}\right)$$

where

$$f(\hat{x}) \equiv g^{-1/(\alpha+1)}(y)$$

which provides the link to the results of section 3.

Table 2. A generalization of table 1 to include an arbitrary function $g(y)$.

$$\begin{tikzcd}[column sep=small]
\begin{array}{l}(1)\\ u_t = \dfrac{1}{\alpha+1}[g(y)u^{\alpha+1}]_{yy}\end{array} \arrow[r, leftrightarrow, "u=v_y"] \arrow[ddd, leftrightarrow, dashed, "\begin{array}{l}\rho=-y^3u\\ \xi=1/u\end{array}"] & \begin{array}{l}(2)\\ v_t = \dfrac{1}{\alpha+1}[g(y)v_y^{\alpha+1}]_y\end{array} \arrow[r, leftrightarrow, "v=w_y"] \arrow[d, leftrightarrow] & \begin{array}{l}(3)\\ w_t = \dfrac{1}{\alpha+1}g(y)w_{yy}^{\alpha+1}\end{array} \arrow[d, leftrightarrow, dashed, "w+z=vy"] \\
 & \begin{array}{l}(4)\\ y_t = -\dfrac{1}{\alpha+1}[g(y)y_v^{-(\alpha+1)}]_v\end{array} \arrow[r, leftrightarrow, "y=z_v"] & \begin{array}{l}(5)\\ z_t = -\dfrac{1}{\alpha+1}g(z_v)z_{vv}^{-(\alpha+1)}\end{array} \arrow[d, leftrightarrow] \\
 & \begin{array}{l}(6)\\ \xi_t = \dfrac{1}{\alpha+1}[h(\xi)(-\xi_z)^{-(\alpha+1)}]_z\end{array} \arrow[r, leftrightarrow, "\xi=v_z"] \arrow[d, leftrightarrow] & \begin{array}{l}(7)\\ v_t = \dfrac{1}{\alpha+1}h(v_z)(-v_{zz})^{-(\alpha+1)}\end{array} \arrow[d, leftrightarrow, dashed, "\varphi+v=\xi z"] \\
\begin{array}{l}(8)\\ \rho_t = -\dfrac{1}{\alpha+1}[h(\xi)(-\rho)^{\alpha+1}]_{\xi\xi}\end{array} \arrow[r, leftrightarrow, "\rho=z_\xi"] & \begin{array}{l}(9)\\ z_t = -\dfrac{1}{\alpha+1}[h(\xi)(-z_\xi)^{\alpha+1}]_\xi\end{array} \arrow[r, leftrightarrow, "z=\varphi_\xi"] & \begin{array}{l}(10)\\ \varphi_t = -\dfrac{1}{\alpha+1}h(\xi)(-\varphi_{\xi\xi})^{\alpha+1}\end{array}
\end{tikzcd}$$

(c) The relationships between (4), (5), (6) and (7) correspond to the results of subsection 4.3.

(d) The table may be slightly extended without increasing the complexity when $\alpha = -\frac{2}{3}$; by a hodograph transformation (3) then goes to

$$y_t = -3g(y)(-y_{ww})^{1/3}$$

while (10) goes to

$$\xi_t = 3h(\xi)(\xi_{\varphi\varphi})^{1/3}.$$

Similar extensions apply to table 1.

(e) A number of exactly linearizable equations are obtained by setting $\alpha = 0$.

(f) The two sets of equations (1)-(5) and (6)-(10) are essentially identical if g satisfies

$$g(\xi) = \xi^{3\alpha+4} g(1/\xi)$$

so that

$$h(\xi) = g(\xi).$$

This condition is met if

$$g(\xi) = \xi^{(3/2)\alpha+2} G(\xi) G(1/\xi)$$

for any function G.

We may slightly generalize the classes of exactly linearizable equations which we have determined. We noted in section 3 that equations of the form

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x}\left[h(v)\left(\frac{\partial v}{\partial x}\right)^{-1}\right]$$

are exactly linearizable and subsection 5.1 showed that the same holds for

$$\frac{\partial v}{\partial t} = \left(\frac{\partial v}{\partial x}\right)^{-2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial x}(\varphi(v)).$$

These results may be combined and generalized by considering equations of the form

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[-h(v)\left(\frac{\partial v}{\partial x}\right)^{-1} + \varphi(v)\right] \tag{7.1}$$

which maps to

$$\frac{\partial V}{\partial T} = \frac{\partial}{\partial X}\left(h(X)\frac{\partial V}{\partial X} - \varphi(X)\right)$$

and hence to

$$\frac{\partial U}{\partial T} = \frac{\partial^2}{\partial X^2}[h(X)U - \varphi(X)] \tag{7.2}$$

so that (7.1) is linearizable for any functions h and φ. Equation (7.1) also integrates to

$$\frac{\partial w}{\partial t} = -h\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^{-1} + \varphi\left(\frac{\partial w}{\partial x}\right) \tag{7.3}$$

which transforms to

$$\frac{\partial W}{\partial T}=h(X)\frac{\partial^2 W}{\partial X^2}-\varphi(X)$$

so that equations of the form (7.3) are also exactly linearizable.

These results may be further generalized slightly to, for example, equations of the form

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left[-h(v)\left(\frac{\partial v}{\partial x}\right)^{-1}+\varphi(v)\right]+\psi(v).$$

This maps to

$$\frac{\partial V}{\partial T}=\frac{\partial}{\partial X}\left(h(X)\frac{\partial V}{\partial X}-\varphi(X)\right)-\psi(X)\frac{\partial V}{\partial X}$$

which is linear for any functions h, φ and ψ.

We also note that if we choose

$$D^*(U)=U^{-2} \qquad \Phi(X)=X$$

then (5.9) is Burgers' equation, so that

$$\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(U^{-2}\frac{\partial U}{\partial X}\right)-1 \tag{7.4}$$

is exactly linearizable. By choosing

$$E\left(u,\frac{\partial u}{\partial x}\right)=u^{-2}\frac{\partial u}{\partial x}$$

in (5.17), equation (5.18) is linear, so that

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right)+\alpha x\frac{\partial u}{\partial x}+\beta u \tag{7.5}$$

is also exactly linearizable. The linearizability of (7.4) and (7.5) was noted by Svinolupov [31].

Some extensions of our results to higher-order equations (cf [28]) and to higher dimensions are possible. Considering, for example, the two-dimensional version of (1.3):

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}[K(u)]+\frac{\partial^2}{\partial y^2}[K(u)]$$

then writing

$$u=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$$

gives without loss of generality

$$\frac{\partial w}{\partial t}=K\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$$

and the two-dimensional form of the Legendre transformation is as follows:

$$w=X\frac{\partial W}{\partial X}+Y\frac{\partial W}{\partial Y}-W \qquad x=\frac{\partial W}{\partial X} \qquad y=\frac{\partial W}{\partial Y} \qquad t=T$$

and yields

$$\frac{\partial W}{\partial T} = -K \left[\frac{\dfrac{\partial^2 W}{\partial X^2} + \dfrac{\partial^2 W}{\partial Y^2}}{\dfrac{\partial^2 W}{\partial X^2}\dfrac{\partial^2 W}{\partial Y^2} - \left(\dfrac{\partial^2 W}{\partial X\,\partial Y}\right)^2} \right]. \tag{7.6}$$

However, the physical relevance of equations of the form (7.6) is not clear.

Many of our results can be extended by combining the non-local transformations with local ones, as illustrated in subsection 4.4. For example, in equation (7.2) φ may without loss of generality be set to zero by replacing U by

$$U - \left(\frac{\varphi(X)}{h(X)}\right) \tag{7.7}$$

so that φ may also be set to zero in equation (7.1) by a non-local transformation based on (7.7) in which x is replaced by

$$x - \int^{v} \frac{\varphi(\hat{v})}{h(\hat{v})}\,\mathrm{d}\hat{v}.$$

As already noted, many of our results are particularly useful in dealing with equations of the form

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[h(v)\left(\frac{\partial v}{\partial x}\right)^n \right] \tag{7.8}$$

see, in particular, (4) and (18) of table 1 and (4) and (6) of table 2. More detailed results for (7.8) will be presented elsewhere.

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